CALCULUS II
THE INTEGRAL OF HYPERBOLIC SINE

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We compute the integral of the hyperbolic sine function in three ways, and compare the results.

Method 1: Using the substitution $u = \cosh x$:

$$\begin{aligned}
\int \sinh x \, dx &= \int \frac{dx}{\cosh x} \\
&= \int \frac{\sinh x \, dx}{\cosh x \sinh x} \\
&= \int \frac{\sinh x \, dx}{\cosh x \sqrt{\cosh^2 x - 1}} && \text{via the identity } \cosh^2 x - \sinh^2 x = 1 \\
&= \int \frac{du}{u\sqrt{u^2 - 1}} && \text{where } u = \cosh x \text{ so } du = \operatorname{sech} x \, dx \\
&= \operatorname{arcsec} u + C \\
&= \operatorname{arcsec}(\cosh x) + C.
\end{aligned}$$

Method 2: Using the substitution $u = \sinh x$:

$$\begin{aligned}
\int \sinh x \, dx &= \int \frac{dx}{\cosh x} \\
&= \int \frac{\cosh x \, dx}{\cosh^2 x} \\
&= \int \frac{\cosh x \, dx}{1 + \sinh^2 x} && \text{via the identity } \cosh^2 x - \sinh^2 x = 1 \\
&= \int \frac{du}{1 + u^2} && \text{where } u = \sinh x \text{ so } du = \cosh x \, dx \\
&= \arctan u + C \\
&= \arctan(\sinh x) + C.
\end{aligned}$$

Method 3: Using the substitution $u = e^x$:

$$\begin{aligned}
\int \sinh x \, dx &= \int \frac{2 \, dx}{e^x + e^{-x}} \\
&= \int \frac{2e^x \, dx}{e^{2x} + 1} \\
&= \int \frac{2du}{u^2 + 1} \quad \text{where } u = e^x \text{ so } du = e^x \, dx \\
&= 2 \arctan u + C \\
&= 2 \arctan(e^x) + C.
\end{aligned}$$

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We may conclude from this that the three functions

$$f(x) = \operatorname{arcsec}(\cosh x),\ g(x) = \arctan(\sinh x), \quad \text{and } h(x) = 2\arctan(e^x)$$

have the same derivative

$$f'(x) = g'(x) = h'(x) = \operatorname{sech} x.$$

Therefore any pair of these differ by a constant.

We know $g(x) = f(x) + C$ for some C and all $x \in \mathbb{R}$. For $x = 0$, we have

$$\begin{aligned} \arctan(\sinh 0) = \operatorname{arcsec}(\cosh 0) + C &\Rightarrow \arctan(0) = \operatorname{arcsec}(1) + C \\ &\Rightarrow 0 = 0 + C \\ &\Rightarrow C = 0; \end{aligned}$$

therefore $\arctan(\sinh x) = \operatorname{arcsec}(\cosh x)$.

We know $h(x) = g(x) + C$ for some C and all $x \in \mathbb{R}$. For $x = 0$, we have

$$\begin{aligned} 2\arctan(e^0) = \arctan(\sinh 0) + C &\Rightarrow 2\arctan(1) = \arctan(0) + C \\ &\Rightarrow \frac{\pi}{2} = 0 + C \\ &\Rightarrow C = \frac{\pi}{2}; \end{aligned}$$

therefore $2\arctan(e^x) = \arctan(\cosh x) + \frac{\pi}{2}$.

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